

# Measuring Progressivity and Inequality

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## Résumé

On définit une classe générale d'indices de progressivité qui est cohérente avec la théorie bien connue de la mesure de l'inégalité et du bien-être social. On démontre, en particulier, que plus un impôt est progressif, plus la distribution des revenus devient égale, et plus nos indices de progressivité sont élevés. Dans le cas d'une fonction de bien-être social additive et d'un impôt progressif, l'indice de progressivité augmente avec le degré d'aversion relative à l'inégalité. On traite aussi du lien entre l'inégalité des revenus bruts et la mesure de la progressivité. On dérive finalement une classe générale de mesures d'inégalité qui ne sont pas affectées par des variations équi-proportionnelles dans les revenus. Nous illustrons le tout à l'aide du système britannique de taxes et de transferts.

Mots-clé: Progressivité, redistribution, inégalité, bien-être social

## Abstract

A general class of progressivity indices is proposed which is consistent with the well-developed theory of the measurement of inequality and social welfare. In particular, we show that the more progressive a tax system, the more equal the distribution of net income and the greater the progressivity index. For an additive social welfare function and a progressive tax system, the greater the degree of relative inequality aversion, the greater the progressivity index. We also discuss the link between inequality of gross income and tax progressivity. A by-product is the derivation of a general class of inequality measures that are invariant to equi-proportionate changes in incomes. We illustrate the analysis using the British tax and benefit system.

**JEL Number:** D31, D63, H23

**Keywords:** Progressivity, redistribution, inequality, social welfare

## **1-Introduction**

Inequality reduction is surely one of the predominant justifications for the progressivity of taxes and benefits<sup>1</sup>. We therefore propose here a general method for the measurement of progressivity which is consistent with the well-developed theory of inequality measurement. This is in line with the early perspective of Musgrave and Thin (1948) who proposed the concept of "effective progression" to measure "the extent to which a given tax structure results in a *shift in the distribution of income toward equality*" (p.510). In particular, we seek progressivity indices which imply that the greater the measured progressivity of a tax system, the more equal the associated net income distribution. Furthermore, since it is clear, as Atkinson (1970) noted, that "*any measure of inequality involves judgements about social welfare*" (p.257), the same should naturally be true for measures of progressivity. Building on the contribution of Blackorby and Donaldson (1984), our general method makes this dependence explicit.

We proceed by presenting an indicator that measures the performance of taxes and benefits in generating a higher level of welfare than an equal-yield proportional tax or benefit. An index of progressivity is then defined as a function of this performance indicator. We show that the more progressive the tax system, the greater its performance, and the greater its progressivity index. For a progressive tax or benefit, the more inequality averse the social welfare function, the greater its assessed performance and the greater the value of the progressivity index. This reflects the dependence of any progressivity measure upon social welfare judgements. We also present sufficient conditions for which the progressivity index of a progressive tax or benefit system is increasing in the inequality of the gross income distribution.

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<sup>1</sup> See, for instance, the introduction of the remarkable book of Vickrey (1947): "Progressive taxation may be defined as taxation which tends to promote economic equality (i.e., a more equal distribution of income, wealth, consumption, or other measure of economic status)" (p.3). Note also the forceful link between progressivity and equality made by Blum and Kalven (1953, p.70) in their classic treatment of progressive taxation:

"However uncertain other aspects of progression may be, there is one thing about it that is certain. A progressive tax on income necessarily operates to lessen the inequalities in the distribution of that income. In fact, as was noted at the outset, progression cannot be defined meaningfully without reference to its redistributive effect on wealth or income. It would seem therefore that any consideration of progression must at some time confront the issue of equality."

Section 2 introduces some notation and defines an index of the performance of an income tax relative to the performance of a proportional tax. In Section 3, we show that the performance indicator of a class of tax systems with constant residual progression can be interpreted as an index of inequality in the distribution of gross incomes that is invariant to equi-proportionate income changes. Section 4 derives the properties of the performance indicator with respect to changes in progressivity, relative inequality aversion, and the inequality of the gross income distribution. From this, we propose in Section 5 a monotonic function of the performance indicator as a general index of progressivity.

The progressivity indices proposed by this general method contrast with some existing indices in two major ways. First, the dependence of existing indices upon social welfare attitudes is often not clear or explicit<sup>2</sup>. Second, changes in the value of some of these indices fail to be normatively significant. We thus discuss briefly in Section 6 the difference between some existing indices of progressivity and those implied by our general method.

Section 7 indicates how we may develop the proposed indices in two dimensions. First, we account for the separate contribution of individual taxes and benefits to the progressivity of the overall tax and benefit system. Second, we account for the contribution of separate socio-economic groups to the redistribution and progressivity of the tax system over the whole population. Section 8 illustrates the results using the British tax and benefit system, and Section 9 concludes. As is conventional practice in the literature, we assume throughout the analysis that gross incomes are exogenous.

## **2-Definitions**

Define social welfare as a general function  $W_y = W(y_1, \dots, y_H)$  of  $H$  positive individual incomes. Consider the class  $W^C$  of social welfare functions that are increasing, symmetric and quasi-concave<sup>3</sup> in incomes  $y_h$ . These conditions ensure that a mean- and

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<sup>2</sup> See, for instance, Kakwani (1977), Suits (1977), Reynolds and Smolensky (1977), and some of the alternative measures indicated in Pfähler (1983). For a general discussion on this, see Lambert (1993).

<sup>3</sup> A weaker condition than quasi-concavity, Schur-concavity [see Dasgupta, Sen and Starrett (1973)], which also implies symmetry in  $y_h$ , could also be assumed without affecting the results.

rank-preserving transfer from a rich to a poor will not decrease social welfare.

Denote the level  $E_y$  of equally distributed equivalent income<sup>4</sup> as

$$E_y = E \mid W(E, \dots, E) = W(y_I, \dots, y_H) \quad (1)$$

We will sometimes find it notationally more convenient to assume that the income distribution is continuous, with distribution function  $F(\bullet)$  and density  $f(\bullet)$ , and with  $a$  and  $z$  respectively the minimum and maximum income levels.

An index of inequality  $I_y$  in the distribution of  $y$  can then be defined as

$$I_y \equiv 1 - \frac{E_y}{\mu_y}, \quad \text{with } \mu_y = \int_a^z x f_y(x) dx \quad (2)$$

and where the quasi-concavity of  $W$  ensures that  $I_y \in [0,1]$ . By definition, we note that

$$E_y \equiv \mu_y(1 - I_y) \quad (3)$$

Let the distribution of gross incomes  $X$  be  $F_X$  and let the tax schedule (which can be negative) be  $T(X)$ . The distribution of gross income and the tax schedule yield the average tax rate  $t$ ,

$$t = \frac{\int_a^z T(x) f_X(x) dx}{\mu_X} \quad (4)$$

as well as the distribution  $F_N$  of net income  $N(X)=X-T(X)$  and the distribution  $F_{(1-t)X}$  of net income under a yield-equivalent proportional tax  $tX$ . A conventional condition for tax progressivity is

$$\frac{d \left[ \frac{T(X)}{X} \right]}{dX} > 0, \quad \forall X \quad (5)$$

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<sup>4</sup> See Atkinson (1970) and Sen (1973).

To compare the progressivity of different tax regimes, we adopt the Residual Progression (RP) criterion, which is the only local progression measure consistent with welfare and inequality dominance under the Lorenz criterion [Jakobsson (1976)]. RP is the elasticity of net income with respect to gross income and is equal to

$$RP(X) = \frac{d[X-T(X)]}{dX} \cdot \frac{X}{X-T(X)} \quad (6)$$

We can say that  $T_1(X)$  is more progressive than  $T_2(X)$  if and only if  $RP_1(X) < RP_2(X)$ ,  $\forall X$ . If, for a particular  $F_X$ ,  $T_1(X)$  and  $T_2(X)$  generate the same mean net income but  $T_1$  is more progressive than  $T_2$ , then<sup>5</sup>

$$E_{N_1} \geq E_{N_2}, \quad \forall W \in W^C \quad (7)$$

Now compare a progressive tax  $T(X)$  with an equal-yield proportional tax  $tX$ . By (7), we know that welfare under  $T(X)$  cannot be inferior to welfare under  $tX$ . To equalise welfare under the two regimes, we may impose a proportional surcharge tax  $s$  on  $N(X)=X-T(X)$  to yield  $(1-s)N$  and  $E_{(1-s)N}$ . We define as  $\tau_{N,X}$  the unique value of  $s$  which equalises welfare under  $T(X)$  and  $tX$ :

$$E_{(1-\tau_{N,X})N} \equiv E_{(1-t)X} \quad (8)$$

We then have the following implicit function for  $\tau_{N,X}$ :

$$\tau_{N,X} = \frac{I_{(1-t)X} - I_{(1-\tau_{N,X})N}}{1 - I_{(1-\tau_{N,X})N}} \quad (9)$$

We can view  $\tau_{N,X}$  as a "performance" indicator<sup>6</sup> of the tax system  $T(X)$ <sup>7</sup>.

Now let  $W^H$  represent the class of homothetic social welfare functions that belong to  $W^C$ . We know from Blackorby and Donaldson (1978) that any  $W \in W^H$  yields

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<sup>5</sup> See Jakobsson(1976) and Fellman(1976).

<sup>6</sup> The standard relative to which the performance of the tax system  $T(X)$  [yielding  $N(X)=X-T(X)$ ] is assessed here is an equal-yield tax system with proportional taxation,  $tX$ . Alternatively, we could also compare the performance of a tax system whose net proceeds are redistributed proportionally to net income, yielding  $N(X)=[X-T(X)]/(1-t)$ , to the performance of no taxation, maintaining gross incomes  $X$ . For homothetic social welfare functions, the two types of performance indicators would be identical.

<sup>7</sup> For assessing the performance of tax systems with constant residual progression, or for estimating the average residual progression of unevenly progressive tax and benefit systems, see Duclos (1995).

a unique equally-distributed equivalent income function  $E=\mu(1-I)$  with  $I$  homogeneous of degree 0 in income levels. Conversely, we can derive from any index  $I$  of inequality that is homogenous of degree zero in income levels an associated class of homothetic social welfare functions  $W$ . When  $W$  belongs to  $W^H$ , we thus find that  $\tau_{N,X}$  in (9) reduces to

$$\tau_{N,X} = \frac{I_X - I_N}{1 - I_N} \quad (10)$$

### **3-Invariance of Inequality Measures to Proportionate Changes in Incomes**

We pause for a moment to indicate how the performance indicator of a particular class of tax systems can be interpreted as a general class of inequality indices that are invariant to proportional changes in all incomes.

An important standard for tax progressivity is the class of tax systems of the form  $T(X)=X-AX^\theta$ , which exhibit a constant degree  $\theta \in [0,1]$  of residual progressivity<sup>8</sup>. We then have  $N(X)=AX^\theta$ . With this, we can construct an alternative monotonic transformation of the social welfare function  $W$  which gives the average income  $E_{X,\theta}$  of the distribution of  $AX^\theta$  that yields the same welfare as the distribution of  $X$ . In the discrete setting, this gives:

$$E_{X,\theta} = A \theta M_X(\theta) \mid W(A X_1^\theta, \dots, A X_H^\theta) = W(X_1, \dots, X_H) \quad (11)$$

where

$$\theta M_X(\theta) = \frac{1}{H} \sum_{h=1}^H X_h^\theta \quad (12)$$

We have that  $E_{X,0} \equiv E_X$ , which simplifies to the equally distributed equivalent income function found in equation (1). We also note that  $E_{X,1} \equiv \mu_X$ . For  $\theta \in [0,1]$ ,  $E_{X,\theta}$  is increasing in  $\theta$ .

We may then define the following class  $I_{X,\theta}$  of inequality indices:

$$I_{X,\theta} = 1 - \frac{E_{X,\theta}}{\mu_X}, \quad \theta \in [0,1] \quad (13)$$

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<sup>8</sup> For an early use of this system, see Edgeworth (1919) and Vickrey (1947).

$I_{X,\theta}$  measures, as a proportion of average income, the money-metric welfare loss of moving from a distribution  $AX^\theta$  to a distribution  $X$ .

When  $W$  is homothetic, this class of inequality indices has the property of being the most general one from which we may derive inequality orderings that are insensitive to equiproportional changes in incomes  $X$ .

*Proposition 1:*

Let  $E_X^* \equiv \mu_X(1 - I_X^*)$  be any increasing monotonic transformation of  $W$ . Then:

$I_X^* = I_{\lambda X}^*, \forall W \in W^H, \forall F_X$  and  $\forall \lambda > 0 \iff I_X^* = I_{X,\theta}^*$ , for some  $\theta \in [0,1]$  and  $\forall F_X$ .

*Proof:*

$\Rightarrow$  Since  $W \in W^H$ , Lorenz dominance is the appropriate necessary and sufficient criterion by which to judge inequality orderings based on  $I_X^*$  [Dasgupta, Sen and Starrett (1973)]. By Jakobsson (1976), the only inequality standard based on Lorenz curves which is invariant to equiproportional changes  $\lambda$  is the constant RP system,  $AX^\theta$ . Hence, it must be that  $I_X^* = I_{X,\theta}^*$ , for some  $\theta \in [0,1]$ .

$\Leftarrow$  For all homothetic  $W$  we may assess  $I_{X,\theta}$  through a comparison of the Lorenz curves for  $X$  and  $AX^\theta$ . By Jakobsson (1976) and for  $\lambda > 0$ , an identical such assessment must be made from a comparison of  $\lambda X$  and  $A(\lambda X)^\theta$ . Hence, whenever  $I_X^* = I_{X,\theta}^*$  for a given  $\theta$ , it must be that  $I_X^* = I_{\lambda X}^*, \forall W \in W^H, \forall F_X$  and  $\forall \lambda > 0$ .  $\square$

When  $W$  belongs to  $W^H$ , the inequality index  $I_{X,\theta}$  is independent of  $t$  and  $A$ :

$$I_{X,\theta} = \frac{I_X - I_{X^0}}{1 - I_{X^0}} \quad (14)$$

We can easily check that  $I_{X,\theta}$  is precisely the performance index  $\tau_{\theta,X}$  necessary to equalise social welfare under equal mean distributions  $X$  and  $AX^\theta$ :  $\tau_{\theta,X} \equiv I_{X,\theta}$ . Hence, we may see  $I_{X,\theta}$  either as a general index of inequality that is invariant to equi-proportionate changes in  $X$  or as the performance of a tax system with constant residual progression. The Atkinson (1970) and Sen (1973) index of inequality is a particular case of  $\tau_\theta$  and  $I_{X,\theta}$  obtained when  $\theta=0$ . A corollary of Proposition 2 below would be that the greater the value of  $\theta$ , the lower is  $I_{X,\theta}$  and  $\tau_{\theta,X}$ .



#### **4-The Properties of the Performance Indicator**

We now derive some of the general properties of  $\tau_{N,X}$ . For this, we need to assume that the tax system does not rerank individuals. We first show that  $\tau_{N,X}$  is increasing in the progressivity of the tax system.

*Proposition 2:*

For any  $W \in W^H$  and for any  $F_X$ , the greater the progressivity of  $T(X)$ , the lower the indicator  $\tau_{N,X}$ .

*Proof:*

The proof is simple in the light of the Jakobsson(1976)-Fellman(1976) and Dasgupta et al.(1973) theorems linking tax progressivity to Lorenz dominance and Lorenz dominance to changes in inequality as measured by the difference in the relative indices  $I_X - I_N$ . The greater the progressivity of a tax system  $T(X)$ , the closer to the diagonal lies the Lorenz curve of the distribution of net income  $N(X) = X - T(X)$ . This inward movement must (decrease) not increase the value of the unique relative inequality index  $I_N$  of an homothetic, symmetric and (strictly) quasi-concave welfare function  $W$ . Hence, by (10),  $\tau_{N,X}$  must (increase) not fall when the progressivity of  $T(X)$  is increased.  $\square$

A similar result also holds for social welfare functions that are not homothetic, so long as the distributions being compared have the same mean income.

*Proposition 3:*

If  $T_1(X)$  is more progressive than  $T_2(X)$  and, for a given  $F_X$ ,  $\mu_{N1} = \mu_{N2}$ , then, for that  $F_X$ ,  $\tau_{N1,X} \geq \tau_{N2,X}$  for any  $W \in W^C$ .

The proof is analogous to the one of Proposition 2.  $\square$

Let  $W^A$  represent the subclass of social welfare functions that are additive in functions of individual incomes and that belong to  $W^C$ .  $W \in W^A$  then takes the form of

$$W_y = \int_a^z U(x) f_y(x) dx \quad (15)$$

where  $U$  is increasing and concave. Note that  $W \in W^A$  need not be homothetic. Adopt the following standard definition of relative inequality aversion  $A(x)$ :

$$A(x) = \frac{-U''(x) \cdot x}{U'(x)} \quad (16)$$

*Proposition 4:*

For a progressive tax  $T(X)$ , the greater the relative inequality aversion of the function  $U(\bullet)$ , the no smaller can be the performance indicator  $\tau_{N,X}$ .

*Proof<sup>9</sup>:*

Take  $U^*$  to exhibit greater inequality aversion than  $U$ . Let  $A^*$  and  $\tau_{N,X}^*$  be respectively the inequality aversion and the performance indicator corresponding to  $U^*$ . By construction, we have that  $A^*(x) \geq A(x)$ ,  $\forall x > 0$ , with strict inequality holding for some range of  $x$ . By Pratt (1964) we know that  $A^*(x) \geq A(x)$  is equivalent to the existence of a function  $\phi$  such that  $\phi' > 0$ ,  $\phi'' \leq 0$ , and  $U^*(x) = \phi[U(x)]$ ,  $\forall x > 0$ .

Now consider the initial welfare equalising surcharge tax  $\tau_{N,X}$ . By definition, we have that

$$\int_a^z U[(1-\tau_{N,X}) \cdot N(x)] f_X(x) dx = \int_a^z U[(1-t) \cdot x] f_X(x) dx \quad (17)$$

It  $T(X)$  is progressive, we have that

$$\frac{(1-\tau_{N,X}) \cdot N(x)}{(1-t) \cdot x} = \frac{(1-\tau_{N,X})}{(1-t)} \cdot \frac{N(x)}{x} \quad (18)$$

is a decreasing function of  $x$ . Hence, since  $U[(1-\tau_{N,X}) \cdot N(x)]$  has by definition the same mean as  $U[(1-t) \cdot x]$ , it must be that  $(1-\tau_{N,X}) \cdot N(x)$  is greater (lower) than  $(1-t) \cdot x$  for all  $x$  lower (greater) than some  $\hat{x}$ , with  $(1-\tau_{N,X}) \cdot N(\hat{x}) = (1-t) \cdot \hat{x}$ .  $U[(1-\tau_{N,X}) \cdot N(x)]$  is then more equally distributed than the equal mean distribution of  $U[(1-t) \cdot x]$ . Applying the Atkinson(1970) theorem to the concave function  $\phi$ , we find that

$$\int_a^z \phi \left\{ U[(1-\tau_{N,X}) \cdot N(x)] \right\} f_X(x) dx \geq \int_a^z \phi \left\{ U[(1-t) \cdot x] \right\} f_X(x) dx \quad (19)$$

Hence, it must be that  $\tau_{N,X}^* \geq \tau_{N,X}$ .  $\square$

If the condition in Proposition 4 were "the greater *everywhere* the relative inequality aversion of the function  $U(\bullet)$ ", with the strict inequality  $A^*(x) > A(x)$ ,  $\forall x > 0$ , we would find that  $\tau_{N,X}^* > \tau_{N,X}$  when  $T(X)$  is progressive.

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<sup>9</sup> An alternative and more intricate proof that does not use the Pratt (1964) result can be found in the Appendix.

In a well-known paper, Musgrave and Thin (1948) expressed the following conjecture: "The less equal is the distribution of income before tax, the more potent will be a progressive tax structure in equalizing income" (p.510). In a different context and framework, Suits (1977) suggests that:

"There is nothing inherently regressive about a sales tax or even a poll tax.

They are regressive because income is unequally distributed, and the more unequally income is distributed, the more regressive they become." (p.752)

It is interesting to note some of the conditions under which their presumption holds when the equalising power of income tax progressivity is judged by the Lorenz criterion.

Let a disequalising but mean-preserving shift occur to the distribution of gross incomes, causing a movement from distribution X to distribution X'. This could take the form, say, of a transfer from a poor i to a rich j that does not affect the ranking of the two units. Average gross income is maintained, but average net income can either increase, decrease or stay the same, depending on the difference  $T'_i - T'_j$  between the marginal tax rates of the two individuals involved in the transfer. This is so since the progressivity of a tax system does not imply that marginal tax rates must move in any particular direction across the income distribution (except that they must remain above the average tax rate).

Assume, however, that  $T'_i < T'_j$ ; the disequalising shift will therefore decrease average net income by  $(T'_j - T'_i)/H$ . Figure 1 then shows how the Lorenz curves for X and N shift to the Lorenz curves for X' and N'. A Lorenz curve for  $L_y$  is defined as

$$L_y\left(\frac{k}{H}\right) = \sum_{h=1}^k \frac{y_h}{H\mu_y}, \quad k=1, \dots, H, \quad \text{and with } L(0)=0 \quad (20)$$

On Figure 1,  $L_X$  is nowhere below  $L_{X'}$  and lies above it in the interval that separates the poor i and the rich j between which the disequalising transfer occurred. Because we have assumed that the average net income was decreased by the disequalising transfer,  $L_N$  crosses  $L_{N'}$  once from below.

*Proposition 5:*

Let  $W \in W^H$  and X' and N' be the distributions of gross and net incomes when a shift of income that preserves the ranks and the mean but increases the spread occurs between the gross incomes of individual i and j. Then

$$\tau_{N',X'} \geq \tau_{N,X} \quad (21)$$

if  $T'_i < T'_j$  and if at least one of the following conditions is fulfilled:

- (a)  $W$  is sufficiently inequality averse in the income shares of the least well-off;
- (b)  $\sigma_N > \sigma_{N'}$  (where  $\sigma_y$  is the standard deviation of  $y$ ) and  $W$  belongs to  $W^A$  and obeys the principle of diminishing transfers<sup>10</sup>.

*Proof:*

$\tau_{N',X'} \geq \tau_{N,X}$  is equivalent to requiring that

$$\frac{\tau_{N',X'}}{1-\tau_{N',X'}} = \frac{I_{X'}-I_{N'}}{1-I_{X'}} \geq \frac{I_X-I_N}{1-I_X} = \frac{\tau_{N,X}}{1-\tau_{N,X}} \quad (22)$$

Rearranging, we find that this condition can be rewritten as:

$$\frac{\tau_{X,X'}}{1-\tau_{X,X'}} = \frac{I_{X'}-I_X}{1-I_{X'}} \geq \frac{I_N-I_{N'}}{1-I_{N'}} = \frac{\tau_{N,N'}}{1-\tau_{N,N'}} \quad (23)$$

Inspection of Figure 1 confirms that  $\tau_{X,X'} \geq 0$ .

If (a) holds, then  $\tau_{N,N'} < 0$  and  $\tau_{N',X'} \geq \tau_{N,X}$  also holds.

Because the Lorenz curve for  $N'$  crosses once from above the Lorenz curve for  $N$ , when (b) holds<sup>11</sup>, it must be that  $\tau_{N,N'} \leq 0$ . Hence  $\tau_{N',X'} \geq \tau_{N,X}$ .  $\square$

Proposition 5 indicates that, under some relatively weak sufficient conditions, the performance of a progressive tax system will grow if the distribution of gross incomes becomes less equal. It is, however, also relatively easy to find instances in which this result cannot hold. One such instance would involve a progressive tax system with  $T'_i > T'_j$  and a social welfare function  $W$  that is very inequality averse in the income shares of the least well-off. We would then have in Figure 1 a Lorenz curve for  $N'$  that would cross once from *below* the Lorenz curve for  $N$ , with the result that  $\tau_{N,N'} > \tau_{X,X'}$ , and with  $\tau_{N',X'} < \tau_{N,X}$ . A disequalising shift in the distribution of gross incomes then causes the share of the net income of the least well-off to fall and the performance of the progressive tax system falls. In such circumstances, Musgrave's and Thin's conjecture fails to hold.

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<sup>10</sup> For  $W \in W^A$ ,  $W$  obeys the principle of diminishing transfers if and only if  $U'''(x) > 0$ ,  $\forall x > 0$ . This corresponds to a greater valuation of "a transfer between persons with a given income difference if these incomes are lower than if they are higher" [Kolm(1976), p.87].

<sup>11</sup> See, for instance, Lambert (1993), p.76.

## **5- A General Index of Progressivity**

For notational convenience, we now define an index of equality  $e_x$  as  $e_x = 1 - I_x$ . For homothetic social welfare functions, we have that  $e_x = (1 - \tau_{N,X})e_N$ . An index of progressivity  $\Pi_T$  can then be defined as

$$\Pi_T = \frac{\tau_{N,X}}{1 - \tau_{N,X}} = \frac{e_N - e_x}{e_x} \quad (24)$$

and measures the percentage change in equality induced by taxation. A proportional tax regime will have  $\tau_{N,X} = 0$  and  $\Pi_T = 0$ ; the greater the performance indicator  $\tau$ , the greater the progressivity index  $\Pi_T$ .

For homothetic social welfare functions, values of  $\tau_{N,X}$  and  $\Pi_T$  are easily linked to changes in social welfare when mean incomes vary. Since  $E_x = \mu_x \cdot e_x$  and  $E_N = (1 - t)\mu_x \cdot e_N$ , we have that

$$\frac{E_N - E_x}{E_N} = \frac{1}{1 - t} \cdot (\tau_{N,X} - t) \quad (25)$$

or that

$$\frac{E_N - E_x}{E_x} = \frac{\tau_{N,X} - t}{1 - \tau_{N,X}} = \Pi_T - \frac{t}{1 - \tau_{N,X}} \quad (26)$$

Social welfare will therefore increase under the tax  $T(X)$  if  $\tau_{N,X}$  exceeds the average rate of tax  $t$  or, equivalently, if progressivity  $\Pi_T$  is sufficiently large to compensate for the fall in average income incurred through taxation.

Some well-known indices of progressivity and redistribution can straightforwardly be derived as particular members of the class of indices  $\Pi_T$ . If, for instance, we consider the generalised Gini indices of inequality  $G_x$  proposed by Yitzhaki(1983) and Donaldson and Weymark (1980), which are defined by:

$$G_x(v) = v \int_a^z \frac{(\mu_x - x)}{\mu_x} [1 - F_x(x)]^{v-1} f_x(x) dx, \quad v > 1 \quad (27)$$

we find the corresponding level of equally distributed equivalent income to be:

$$E_x(v) = v \int_a^z x [1 - F_x(x)]^{v-1} f_x(x) dx, \quad v > 1 \quad (28)$$

and that

$$\frac{\tau_{N,X}(v)}{1-\tau_{N,X}(v)} = \frac{G_X(v) - G_N(v)}{1 - G_X(v)} \quad (29)$$

This generalises a normalisation of the well-known Reynolds-Smolensky (1977) index of reduction in the Gini coefficient of inequality, for which  $v=2$ . We also have that

$$\frac{1}{1-\tau_{N,X}(v)} = \frac{1-G_N(v)}{1-G_X(v)} \quad (30)$$

generalises the Musgrave and Thin (1948) index of effective progression.

## **6- Comparisons with Some Other Indices of Progressivity**

Kakwani (1977) and Suits<sup>12</sup> (1977) have proposed progressivity indices that are homogeneous of degree zero in the level of taxes  $T(x)$ , that is, that are invariant to proportional changes in the level of taxation across the income distribution. Though often-used, these measures can exhibit important flaws.

A general class of measures which exhibit these flaws is the general class of *scale invariant aggregate measures of the tax redistribution effect* introduced by Pfähler (1987). The class subsumes the Kakwani and the Suits index. These measures are not defined and cannot be used when  $t=0$ , e.g., whenever the tax system is purely redistributive. Another flaw is that a proportional change in  $T$  can easily make it more or less progressive and more or less inequality-reducing according to equation (6), even though this proportional change will fail to alter the Kakwani or Suits indices. In the terms of Blackorby and Donaldson (1984), these indices are not ethical.

Take, for instance, tax systems  $T_1(x)$  and  $T_2(x)$ , with average local tax rates  $t_i(x)=T_i(x)/x$ ,  $i=1,2$ . Let  $T_1$  be more progressive than  $T_2$  according to the RP criterion of equation (6). This implies that

$$t_1'(x) > \frac{t_2'(x)[1-t_1(x)]}{1-t_2(x)}, \quad \forall x > 0 \quad (31)$$

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<sup>12</sup> See also Hainsworth (1964).

Now multiply the tax schedule  $T_1$  by a factor  $k > 0$ . This linear change leaves intact the progressivity measures of Kakwani and Suits. For  $T_1(x)$  to remain more progressive than  $T_2$  at any point  $x$ , it must, however, be that

$$k > \frac{t_2'(x)}{[1-t_2(x)]t_1'(x) + t_2'(x)t_1(x)}, \quad \forall x > 0 \quad (32)$$

Otherwise,  $T_2$  becomes more locally progressive and may lead to less inequality and more redistribution than  $T_1$ .

An example is provided by the table below, where two taxes, A and B, are alternatively applied onto the same distribution of gross income. Under tax A, net incomes are clearly more equal than under tax B, but the poorer individual, individual 1, bears a lesser share of the total tax burden under tax B than under tax A.

	Individual 1	Individual 2	Total
<b>Gross income</b>	21	80	101
<b>Tax A</b>	1	50	51
<b>Net income under tax A</b>	20	30	50
<b>Tax B</b>	0	1	1
<b>Net income under tax B</b>	21	79	100

Which of tax A or B is then the more progressive? Any measure of progressivity that ties progressivity to inequality reduction would conclude that tax A is the more progressive. Progressivity indices that are homogeneous of degree zero in the level of taxes would, however, necessarily indicate that tax B is the more progressive. To compare progressivity, we find it both sensible and consistent with the tradition of public finance to adopt the first standard, that of inequality reduction. Any consequent measure of progressivity based on that standard must necessarily be consistent with progressivity rankings based on the performance index  $\tau_{N,X}$ , for a particular normative attitude towards inequality and social welfare. Measures that are homogeneous of degree zero in the level

of taxation are then better seen as indices of tax departure from proportionality<sup>13</sup>.

## **7-Extension of the Progressivity Index**

### **7.1-Extension to several taxes**

Let  $T_m$  be tax  $m$ , with  $M$  the total number of taxes. We have that

$$T(X) = \sum_{m=1}^M T_m(X) \quad (33)$$

Denote by  $j$  one of the  $J=M!$  possible orderings of taxes from which we can construct the transition from gross income to net income. We define  $e_{m;j}$  as the equality of income when tax  $T_m$  has just been added to all the taxes that precede  $T_m$  in the ordering  $j$ . Let  $e_{-m;j}$  represent the equality of income just before tax  $T_m$  is added<sup>14</sup>.

We can then define

$$\Pi_{m;j} = \frac{e_{m;j} - e_{-m;j}}{e_X} \quad (34)$$

$\Pi_{m;j}$  is positive if and only if a gain in the performance of the tax system is registered when tax  $T_m$  is added.

For any ordering  $j$ , we then find that:

$$\Pi_T = \sum_{m=1}^M \Pi_{m;j} = \frac{e_N - e_X}{e_X} \quad (35)$$

An alternative index of progressivity  $\Pi_{m;j}^{BD}$  was proposed by Blackorby and Donaldson (1984):

$$\Pi_{m;j}^{BD} = \frac{e_{m;j}}{e_{-m;j}} \quad (36)$$

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<sup>13</sup> Indices of tax departure from proportionality may sometimes help in the construction of progressivity indices consistent with the standard of inequality reduction. For instance, Pfähler (1987) shows that progressivity indices associated with the Mehran (1976) class of linear inequality indices can be expressed as the product of the average rate of taxation with associated indices of tax departure from proportionality.

<sup>14</sup> If, for example, there were  $M=2$  taxes, there could be  $J=2$  possible orderings of taxes. We could have, for instance,  $j=1$  for the case in which  $T_1$  appears first in the construction of net income (giving  $e_{1;1}$ ), followed by  $T_2$  (giving  $e_{2;1}$ ). For  $j=2$ ,  $T_2$  appears first in the ordering, yielding  $e_{2;2}$ , followed by  $T_1$ , yielding  $e_{1;2}$ . We would then have  $e_{-2;1} \equiv e_{1;1}$  and  $e_{-1;2} \equiv e_{2;2}$ .



For any ordering  $j$ , we then have:

$$\Pi_T^{BD} = \prod_{m=1}^M \left( \Pi_{m,j}^{BD} \right) = \frac{e_N}{e_X} \quad (37)$$

We can check that  $\Pi_T^{BD} = 1 + \Pi_T$ .

We may be concerned that the progressivity  $\Pi_{mj}$  of tax  $T_m$  generally depends on the combination of taxes (implied by the ordering  $j$ ) to which  $T_m$  is added. Because of this, we may wish to average  $\Pi_{mj}$  over all  $J$  possible orderings to yield an average measure of progressivity,  $\Pi_m$ , for tax  $T_m$ :<sup>15</sup>

$$\Pi_m = \frac{1}{J} \sum_{j=1}^J \Pi_{m,j} \quad (38)$$

This also implies that

$$\Pi_T = \sum_{m=1}^M \Pi_m \quad (39)$$

## **7.2-Additivity in the Welfare of Subgroups**

If, besides being homothetic, our social welfare function is additive in the welfare of any set of exclusive subgroups of the population, it is well-known that it must take the form of  $W=M(1-\epsilon)$ , where

$$W_y = M_y(1-\epsilon) = \begin{cases} \frac{1}{(1-\epsilon)} \int_a^z x^{(1-\epsilon)} f_y(x) dx, & \text{if } \epsilon \neq 1 \\ \int_a^z \ln x f_y(x) dx, & \text{if } \epsilon = 1 \end{cases} \quad (40)$$

Let  $G$  subgroups divide completely and exclusively the population. Denote by  $f_y^g(x)$  the density of the observations in a subgroup  $g$ , by  $p_g$  the share of the population found in subgroup  $g$ , and by  $M^g(1-\epsilon)$  the welfare of subgroup  $g$  considered as a group. We have that

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<sup>15</sup> A similar averaging process can be applied to  $\Pi_{mj}^{BD}$ .

$$\sum_{g=1}^G p_g = 1 \quad (41)$$

$$f(x) = \sum_{g=1}^G p_g f^g(x) \quad (42)$$

and

$$W_y = M_y(1-\varepsilon) = \sum_{g=1}^G p_g M_y^g(1-\varepsilon) \quad (43)$$

We note that the progressivity of the tax system equals

$$\Pi_T = \frac{e_N - e_X}{e_X} = \left[ \frac{\sum p_g M_N^g}{M_X} \left( \frac{\mu_X}{\mu_N} \right)^{1-\varepsilon} \right]^{1/1-\varepsilon} - 1 \quad (44)$$

Transforming the homothetic functions  $M_y$  as  $(\mu_y e_y)^{1-\varepsilon}$ , we find:

$$\Pi_T = \left\{ \sum p_g \left[ \frac{\mu_X^g (1-t^g)}{\mu_X (1-t)} \frac{e_X^g}{e_X} \frac{e_N^g}{e_X^g} \right]^{1-\varepsilon} \right\}^{1/1-\varepsilon} - 1 \quad (45)$$

where  $\mu_X^g$  and  $t^g$  are respectively the average gross income and the average rate of tax found for the units of group  $g$ . Rearranging, we finally have:

$$\Pi_T = \left\{ \sum p_g \left[ \frac{\mu_X^g (1-t^g)}{\mu_X (1-t)} \gamma^g \left( 1 + \Pi_T^g \right) \right]^{1-\varepsilon} \right\}^{1/(1-\varepsilon)} - 1 \quad (46)$$

where

$$\gamma^g = \frac{e_X^g}{e_X} \quad (47)$$

and

$$\Pi_T^g = \frac{e_N^g - e_X^g}{e_X^g} \quad (48)$$

This decomposition for  $\Pi_T$  takes into account the population weights, the average gross income of group  $g$  relative to the overall mean income, the tax rate  $t^g$  relative to  $t$ , the equality of gross incomes in group  $g$  relative to equality in the overall population, and tax progressivity within group  $g$ . Ceteris paribus, (1) the greater the correlation between group progressivity and average net income of a group  $g$ , or (2) the greater the correlation between group progressivity and population share, then the greater the progressivity of the tax system. The product  $\gamma^g(1+\Pi_T^g)$  indicates that the achievement of a given level of progressivity  $\Pi_T^g$  will have the greatest overall impact if it has been obtained from an initially relatively equal group distribution.

For  $\varepsilon=0$ , we have that  $\Pi_T=0$ . As  $\varepsilon$  increases,  $\Pi_T$  takes more and more account of those groups that include the least well-off members of the population, and it measures progressivity more and more according to the redistributive properties of  $T(X)$  towards those least advantaged members. As  $\varepsilon$  tends to infinity, we have

$$\lim_{\varepsilon \rightarrow \infty} \Pi_T = \frac{\frac{\min\{N\}}{\mu_N} - \frac{\min\{X\}}{\mu_X}}{\frac{\min\{X\}}{\mu_X}} \quad (49)$$

by which progressivity equals the proportional increase in the share of the least well-off member of the population.

## **8- Illustration**

We now illustrate the application of the above results using the 1985 British tax and benefit system. Our sample of 4471 families is drawn from the 1985 Family Expenditure Survey (FES). The FES is a continuous enquiry into the income and characteristics of private households in the United Kingdom. To compute the tax liabilities and benefit entitlements of the families, we make use of the tax and benefit model described in detail in Duclos (1992). For the purposes of our illustration, we do not model the benefits granted in kind by the state, such as those provided by health and education expenditures, nor do we include general housing subsidies, local authority taxes or indirect taxes, or the implicit subsidies or taxes generally implied by various government regulations and activities. The computations do, however, account for the major components of the personal income tax system and of the social security and state benefit system. Here, we regroup these taxes and benefits into four categories:

A- Various benefits: including child benefit, one-parent benefit, unemployment benefit, National Insurance Basic Pension, and diverse disability and injury benefits; these various benefits average 13% of gross income.

B- Personal income taxation: the personal income tax system (which features personal tax allowances and marginal tax rates ranging from 30% to 60%, but net of mortgage interest tax relief) and National Insurance Contributions; these taxes amount to 28% of gross income.

C- Family Income Supplement and Supplementary Benefit (net of housing subsidy component), averaging 2% of gross income.

D- Housing dependent benefits, that is, Housing Benefit (for relief on rents and local property taxes), mortgage interest tax relief, and the housing subsidy element of Supplementary Benefit, representing 7% of gross income.

To adjust for varying household composition and size, we use the equivalence scale implicit in the scale of the main social assistance programme, Supplementary Benefit. Family weights reflect family size and a statistical adjustment for differential population representation in the FES sample. Gross income includes, among other things, labour income, interests and dividends, receipts from private pensions, statutory sick pay, and allowances from friends and former spouses. A more proper basis on which to assess the progressivity of taxes and benefits would also incorporate, for instance, imputed rent from owner-occupied housing and accrued capital gains. In the presence of sound capital markets, we might also rather be concerned with redistribution and progressivity over the lifetime.

We start by illustrating the decomposition of total progressivity into the contribution of separate taxes and benefits. As in equation (38), the separate contribution of a tax  $m$  is the average over all possible  $j$  orderings of the  $\Pi_{mj}$  defined in (34). For expositional clarity, we normalise  $\Pi_m$  by  $\Pi_T = \sum_m \Pi_m$ , the progressivity of the sum of the taxes and benefits. On Figure 2, we show  $\Pi_m/\Pi_T$  on the left axis for the four categories of taxes and benefits defined above. We then depict  $\Pi_T$  along the right axis.

Figure 2 indicates that, as expected, we value the progressivity of the total set of taxes and benefits more and more as inequality aversion increases, and these valuations imply a performance indicator  $\tau_{N,X}$  that increases to 0.81 when  $\epsilon=0.95$ . The contribution of categories A to D of taxes to total progressivity varies generally with  $\epsilon$ . As a proportion of total progressivity, the progressivity of benefits A first increases and then

falls to about a third. In spite of their low average size (2% of gross income), benefits C -- whose grant is specifically targeted towards the low income population -- contribute a respectable and increasing portion of total progressivity. A similar comment applies to the somewhat more important (7% of gross income) housing-dependent benefits. For low values of the relative inequality aversion parameter, personal taxes B contribute almost one fourth of total progressivity, but that contribution falls to 5% as  $\epsilon$  reaches 0.95.<sup>16</sup> Taxes B are thus first deemed more progressive than benefits C and D, but they end up being assessed as the least progressive of all tax and benefit categories. Whatever the value of  $\epsilon$ , however, the sum of the progressivity of transfers (A,C and D) exceeds that of taxes (B).

Figure 3 shows how the progressivity of the tax and benefit system over separate population subgroups,  $\Pi_T^\epsilon$ , varies with values of the inequality aversion parameter  $\epsilon$  ranging between 0.05 and 0.95. For this, we divide the population first into younger families and older families (for which at least one of the members is aged 60 or more). For younger families, we then have families without children, two-adult families with at least one child, and one-adult families with at least one child. Figure 3 shows that  $\Pi_T^\epsilon$  is everywhere above zero for all groups, and thus that the tax and benefit system is deemed progressive for all groups and for all values of  $\epsilon$ . Its assessed degree of progressivity, however, increases with  $\epsilon$ , with corresponding increases in the performance indicator  $\tau_{N,X}$  measuring the average surtax rate which could be imposed to bring welfare to its level under proportional taxation. The index of progressivity over the whole population lies between the index of progressivity for the group of two-adult families with children, which is everywhere the lowest, and the index for the three other groups.

Table 1 illustrates the decomposition of equation (46), where tax and benefit progressivity over the whole population is split into tax progressivity over separate population subgroups; for this, we choose  $\epsilon=0.5$ . Line A of Table 1 shows that 31% of individuals live in younger families without children, 44% in two-adult families with children, and the rest in one-parent (4%) and older families (22%). Line B indicates the ratio of average group income to average population income. Young families without children are, on average, the better off, with older and one-parent families the poorest. The movement from line B (gross income) to line C (net income) exhibits the level of

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<sup>16</sup> This is notwithstanding the feature that  $\Pi_m$  for taxes B does increase, despite its rapid fall relative to  $\Pi_T$ .

redistribution across groups. Average income of young families without children falls by 10% relative to the population average, while the same figure for one-parent and older families almost doubles.

Equality of gross income within groups, relative to gross income equality for the whole population, is shown on line D of Table 1. The least equally distributed group incomes are those of one-parent and older families, a reflection of the large relative variations in the gross incomes of the members of these families. The assessed group progressivity  $\Pi_T^g$  of the tax and benefit system is in inverse relation to gross income inequality, as displayed on line E. Thus, the less equally distributed gross income seemed to be in a group, the more progressive and the more performant the British tax and benefit system seems to be for that group. This is not surprising in the light of the results of Figure 2, which revealed that transfers were the most progressive and the most redistributive. Such transfers affect disproportionately older and one-parent families. When we multiply gross income equality by one plus the percentage change in group inequality,  $(1+\Pi_T^g)$ , we obtain  $\gamma^g(1+\Pi_T^g) = e_N^g/e_X$ , which is the ratio of group equality of net income to population equality of gross income. Line F shows that, through the application of different group progressivity, the tax and benefit system succeeds in bringing equality of net income within a group to approximately the same value for all groups.

Line G weights this ratio of net income equality to population gross income equality with the ratio of net income group averages to net income population average (line C) and with population share (line A). Because equality of net income is approximately the same for all groups, line G is roughly proportional to population share and the ratio of line C. When we normalise those weighted contributions of group progressivity by  $(1+\Pi_T)^{1-\epsilon}$ , we find terms which must sum to one. These terms are shown on line H, and we may consider them as the weighted contribution of group progressivity indices to progressivity over the whole population. Because the ratio of net income equality to gross income equality for the population is the same for all groups, we find on line H that the contribution of each group is quite close to the population share of the group.

Figure 4 indicates how those group contributions to overall progressivity vary with values of the ethical parameter  $\epsilon$ . The contributions for the group of two-adult families with children and for the group of one-parent families vary very little with  $\epsilon$ , hovering around 0.43 and 0.04, their respective population share. The contribution of

younger families without children falls steadily as we become more averse to inequality, with an associated increase in the contribution of the group of older families. This is explained by the greater importance of redistributive benefits (e.g., pensions) among older families, relative to younger families without children who are mostly affected by taxes. As shown on Figure 2, the assessed progressivity of such taxes does not increase so much with the inequality aversion parameter. Thus, the more inequality averse we become, the more we value the progressivity of transfers relative to taxes, and the more overall progressivity is measured through the groups relatively more affected by such transfers.

## **9- Conclusion**

We propose a general class of progressivity indices  $\Pi$  that is consistent with the well-developed theory of the measurement of inequality and social welfare. The tax system is deemed progressive (regressive) when  $\Pi > 0$  ( $\Pi < 0$ ) and equivalent to a proportional tax system when  $\Pi = 0$ . The progressivity indices are a function of a money-metric measure of the superiority (or inferiority) of an income tax regime over an equal-yield, proportional tax system. A simple transformation of the progressivity indices,  $\Pi/(1+\Pi)$ , yields the size of an average surcharge tax which the state could levy while bringing social welfare to its level under proportional taxation. In addition, we show that the more progressive a tax system, the more equal the distribution of net income and the greater the progressivity index. For an additive social welfare function and a progressive tax system, the greater the degree of relative inequality aversion, the greater the progressivity index. We also discuss the link between inequality of gross income and tax progressivity.

A by-product of the analysis is the derivation of a general class of inequality measures that are invariant to equi-proportionate changes in incomes. This class of inequality measures can equivalently be interpreted as the performance of an income tax with constant residual progression. We also show how the proposed progressivity indices can be decomposed into terms accounting for the contribution of separate taxes and benefits to the progressivity of the overall tax and benefit system and into terms accounting for the contribution of separate groups to progressivity over the whole population. We finally illustrate the analysis using the British tax and benefit system.

## Appendix

For an alternative proof of *Proposition 4*, we must first prove the following lemma:

*Lemma A1:*

Define the cumulative difference between the distribution  $F_{(1-t)X}$  and  $F_{(X-T)(1-s)}$  as  $C_s$ :

$$\begin{aligned} C_s(Y) &= \int_a^Y [F_{(1-t)X}(x) - F_{(X-T)(1-s)}(x)] dx \\ C_s(z) &= -s \mu_X(1-t) \end{aligned} \quad (50)$$

For  $T(X)$  progressive, there exists a  $\hat{x}$  such that  $\partial C_\tau(x)/\partial x \geq 0$  for  $x \leq \hat{x}$  and  $\partial C_\tau(x)/\partial x \leq 0$  for  $x \geq \hat{x}$ .

*Proof:*

Define  $\hat{x}$  such that  $[\hat{x} - T(\hat{x})](1-\tau) = \hat{x}(1-t)$ . At  $\hat{x}$  the net income derived from the progressive tax schedule and the performance tax  $\tau$  thus equals the net income under the proportional tax  $t$ . The average tax rate  $t_x$  at  $x = \hat{x}$  then equals  $[1-(1-t)/(1-\tau)]$ . By the progressivity of  $T(x)$ , we have  $t_a < [1-(1-t)/(1-\tau)]$  and that  $t_z > [1-(1-t)/(1-\tau)]$ . It must also be that  $t_x > [1-(1-t)/(1-\tau)]$  for all  $x > \hat{x}$  and that  $t_x < [1-(1-t)/(1-\tau)]$  for all  $x < \hat{x}$ .

Denote  $\Delta GL_\tau(p)$  as the difference between the generalised Lorenz curves for the distributions of  $[X - T(X)](1-\tau)$  and  $X(1-t)$ :

$$\begin{aligned} \Delta GL_\tau(p) &= \int_0^w [x - T(x)] (1-\tau) (1-t) g[x(1-t)] dx \\ &\quad - \int_0^w x(1-t) (1-t) g[x(1-t)] dx \text{ with } p = G[w(1-t)] \end{aligned} \quad (51)$$

The derivative of  $\Delta GL_\tau(p)$  with respect to  $w$  is simply the difference  $[w - T(w)](1-\tau) - w(1-t)$  times the density  $(1-t) \cdot g[w(1-t)]$ . Rearranging terms, we then find:

$$\begin{aligned} \frac{\partial \Delta GL(p)}{\partial w} &= \{ [w - T(w)](1-\tau) - w(1-t) \} (1-t) g[w(1-t)] \\ &= w(1-\tau) \left[ (1-t_w) - \frac{(1-t)}{(1-\tau)} \right] (1-t) g[w(1-t)] \\ &= [t_x - t_w] w(1-t)(1-\tau) g[w(1-t)] \end{aligned} \quad (52)$$



Hence, since, at  $w=\hat{x}$ ,  $\partial\Delta GL_{\tau}(p)/\partial w=0$ , and by the progressivity of  $T(\bullet)$ :

$$\text{for } w \geq \hat{x}, \partial\Delta GL_{\tau}(p)/\partial w \leq 0$$

$$\text{for } w \leq \hat{x}, \partial\Delta GL_{\tau}(p)/\partial w \geq 0$$

This implies that:

$$C_{\tau}(w_1) \geq C_{\tau}(w_0), \text{ for all } w_0 \text{ and } w_1 \text{ such that } w_0 \leq w_1 \leq \hat{x}$$

$$C_{\tau}(w_1) \leq C_{\tau}(w_0), \text{ for all } w_0 \text{ and } w_1 \text{ such that } \hat{x} \leq w_0 \leq w_1$$

and thus that  $\partial C_{\tau}(x)/\partial x \geq 0$  for  $x \leq \hat{x}$  and  $\partial C_{\tau}(x)/\partial x \leq 0$  for  $x \geq \hat{x}$ . It is also apparent from the continuity and differentiability of  $C_{\tau}$  that the strict inequalities  $\partial C_{\tau}(x)/\partial x > 0$  for  $x \leq \hat{x}$  and  $\partial C_{\tau}(x)/\partial x < 0$  must hold for some ranges of  $x$  below and above  $\hat{x}$ .  $\square$

*Proof of Proposition 4:*

For a progressive tax  $T(X)$ , the greater the relative inequality aversion of the function  $U(\bullet)$ , the no smaller can be the performance indicator  $\tau_{N,X}$ .

*Proof:*

Let the general functional form  $A(x)$  denote the relative inequality aversion of the function  $U(x)$ :

$$\frac{-U''(x) \cdot x}{U'(x)} = A(x), \quad x > 0 \quad (53)$$

Integration of the above differential equation yields:

$$U'(x) = k \cdot \exp \left[ \int -\frac{A(x)}{x} dx \right] \quad (54)$$

Consider a step increase  $b$  in the inequality aversion function for  $0 < q < x < r$ , with  $q$  and  $r$  arbitrarily close to each other:

$$\frac{-U''(x) \cdot x}{U'(x)} = \begin{cases} A(x), & \text{if } x \leq q \text{ or } r \leq x \\ A(x)+b, & \text{if } q < x < r \end{cases} \quad (55)$$

We then have:

$$U'(x) = \begin{cases} k \cdot \exp \left[ \int \left( -\frac{A(x)}{x} \right) dx \right], & \text{if } x \leq q \\ k \cdot \exp \left[ \int \left( -\frac{A(x)}{x} - b(\ln x - \ln q) \right) dx \right], & \text{if } q < x < r \\ k \cdot \exp \left[ \int \left( -\frac{A(x)}{x} - b(\ln r - \ln q) \right) dx \right], & \text{if } r \leq x \end{cases} \quad (56)$$

with

$$U''(x) = \begin{cases} \frac{U'(x)}{x} (-A(x)) , & \text{if } x \leq q \text{ or } r \leq x \\ \frac{U'(x)}{x} (-A(x)-b) , & \text{if } q < x < r \end{cases} \quad (57)$$

Taking the derivative of  $U'$  and  $U''$  with respect to  $b$ , we find

$$\frac{\partial U'(x)}{\partial b} = \begin{cases} 0 , & \text{if } x \leq q \\ -(\ln x - \ln q) U'(x) , & \text{if } q < x < r \\ -(\ln r - \ln q) U'(x) , & \text{if } r \leq x \end{cases} \quad (58)$$

and

$$\frac{\partial U''(x)}{\partial b} = \begin{cases} 0 , & \text{if } x \leq q \\ -\frac{U'(x)}{x} - (\ln x - \ln q) U''(x) , & \text{if } q < x < r \\ -(\ln r - \ln q) U''(x) , & \text{if } x \geq r \end{cases} \quad (59)$$

Denote by  $D_s$  the difference between the level of additive social welfare obtained under  $[X-T(X)](1-s)$  and that under  $(1-t)X$ :

$$D_s = \int_a^z U(x) [f_{(X-T)(1-s)}(x) - f_{X(1-t)}(x)] dx \quad (60)$$

where  $U(\bullet)$  is increasing and concave. [By the Jakobsson-Fellman (1976) theorem, we know that  $D_{s=0} \geq 0$ ]. Through double integration by parts of the right-hand side of (62), we can show that:

$$\begin{aligned} D_s &= \int_a^z U'(x) C_s'(x) dx \\ &= -U'(z) C_s(z) - \int_a^z U''(x) C_s(x) dx \end{aligned} \quad (61)$$

where

$$\begin{aligned} C_s(Y) &= \int_a^Y [F_{(1-t)X}(x) - F_{(X-T)(1-s)}(x)] dx \\ C_s(z) &= -s \mu_X(1-t) \end{aligned} \quad (62)$$

as above. Differentiating  $D_s$  with respect to  $b$ , we thus have:

$$\begin{aligned} \frac{\partial D_s}{\partial b} &= -(\ln r - \ln q) U'(z) C_s(z) - \int_q^r \left[ \frac{-U'(x)}{x} - \ln x U''(x) \right] C_s(x) dx \\ &\quad + \int_r^z \ln r U''(x) C_s(x) dx - \int_q^z \ln q U''(x) C_s(x) dx \end{aligned} \quad (63)$$

where  $C'_s(x)$  is the derivative of  $C$  with respect to  $x$ . Integrating by part the first integral on the right, we find:

$$\begin{aligned} \int_q^r \left[ \frac{-U'(x)}{x} - \ln x U''(x) \right] C_s(x) dx &= \\ -U'(r) \ln r C_s(r) + U'(q) \ln q C_s(q) + \int_q^r U'(x) \ln x C'_s(x) dx \end{aligned} \quad (64)$$

Now consider (65) at the initial welfare-equalising  $s=\tau_0$  such that  $D_{\tau_0}=0$ . By (63), we know that

$$U'(z) C_{\tau_0}(z) = \int_a^z U''(x) C_{\tau_0}(x) dx \quad (65)$$

Substituting (66) and (67) into (65) and rearranging, we find

$$\begin{aligned} \left. \frac{\partial D_s}{\partial b} \right|_{s=\tau_0} &= - \int_q^r U'(x) \ln x C'_s(x) dx \\ &\quad + \ln r \int_a^r U'(x) C'_s(x) dx - \ln q \int_a^q U'(x) C'_s(x) dx \end{aligned} \quad (66)$$

Now define  $\hat{x}$  such that  $[\hat{x}-T(\hat{x})](1-\tau)=\hat{x}(1-t)$ . Lemma A1 tells us that  $C'_{\tau_0}(x) \geq 0$  for  $x \leq \hat{x}$  and  $C'_{\tau_0}(x) \leq 0$  for  $x \geq \hat{x}$ . When  $T(X)$  is progressive, it is also apparent that the strict inequalities  $C'_{\tau_0}(X) > 0$  and  $C'_{\tau_0}(X) < 0$  are obtained for at least some range of  $x < \hat{x}$  and  $x > \hat{x}$ ,

respectively. Inspection of (68) reveals that:

$$\left. \frac{\partial D_s}{\partial b} \right|_{s=\tau_0} \geq (\ln r - \ln y) \int_q^r U'(x) C_s'(x) dx + (\ln r - \ln q) \int_a^q U'(x) C_s'(x) dx \quad (67)$$

where  $y = \max\{q, \min\{\hat{x}, r\}\}$

Rearranging,

$$\left. \frac{\partial D_s}{\partial b} \right|_{s=\tau_0} \geq (\ln r - \ln q) \int_a^r U'(x) C_s'(x) dx - (\ln y - \ln q) \int_q^r U'(x) C_s'(x) dx \quad (68)$$

where  $y = \max\{q, \min\{\hat{x}, r\}\}$

By  $D_{\tau_0}=0$ , by the first line of (63), and by Lemma A1, it must be that the first term on the right is no smaller than the second one. Hence,

$$\left. \frac{\partial D_s}{\partial b} \right|_{s=\tau_0} \geq 0 \quad (69)$$

and it must be that  $d\tau/db \geq 0$ .  $\square$

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**Table 1**

**Decomposition of Total Progressivity into the Contributions  
of Four Groups ( $\varepsilon=0.5$ )**

[Index of Total Progressivity:  $\Pi_T(0.5)=0.24$ ]

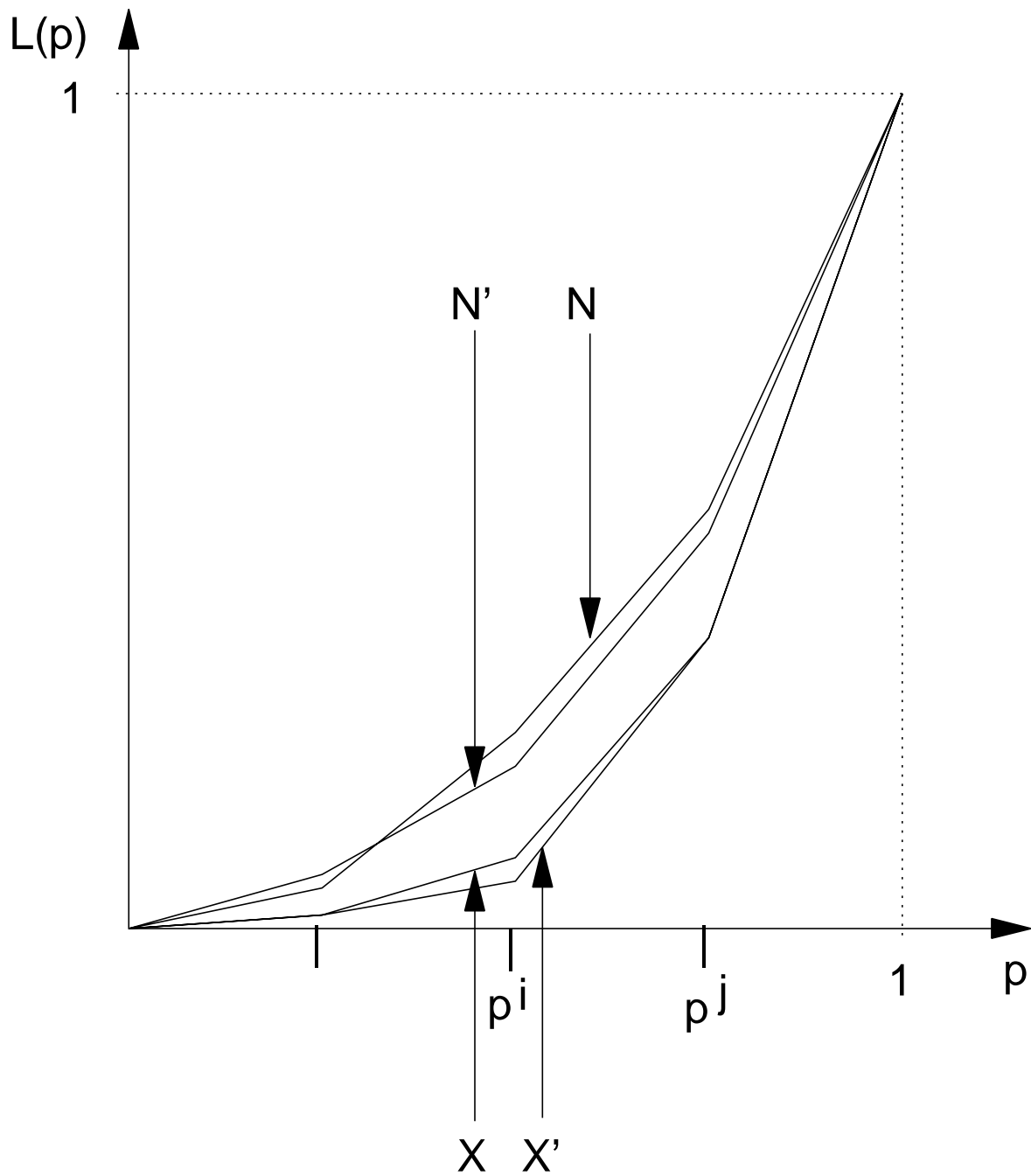
		Young families			Older families
		Without children	Two adults with children	One adult with children	
<b>A</b>	$p_g$	0.31	0.44	0.04	0.22
<b>B</b>	$\frac{\mu_X^g}{\mu_X}$	1.43	1.05	0.38	0.40
<b>C</b>	$\frac{\mu_X^g (1-t^g)}{\mu_X (1-t)}$	1.28	0.99	0.75	0.67
<b>D</b>	$\gamma^g$	1.08	1.13	0.75	0.80
<b>E</b>	$\Pi_T^g$	0.16	0.11	0.71	0.57
<b>F</b>	$\gamma^g (1+\Pi_T^g)$	1.25	1.25	1.27	1.26
<b>G</b>	$p_g \left[ \frac{\mu_X^g (1-t^g)}{\mu_X (1-t)} \gamma^g (1+\Pi_T^g) \right]^{1-\varepsilon}$	0.39	0.49	0.04	0.20
<b>H</b>	$p_g \left[ \frac{\mu_X^g (1-t^g)}{\mu_X (1-t)} \gamma^g (1+\Pi_T^g) \right]^{1-\varepsilon} \frac{1}{(1+\Pi_T)}$	0.35	0.44	0.04	0.18

$$\gamma^g = \frac{e_X^g}{e_X}$$

$$\Pi_T^g = \frac{e_X^g - e_X}{e_X^g}$$

Figure 1

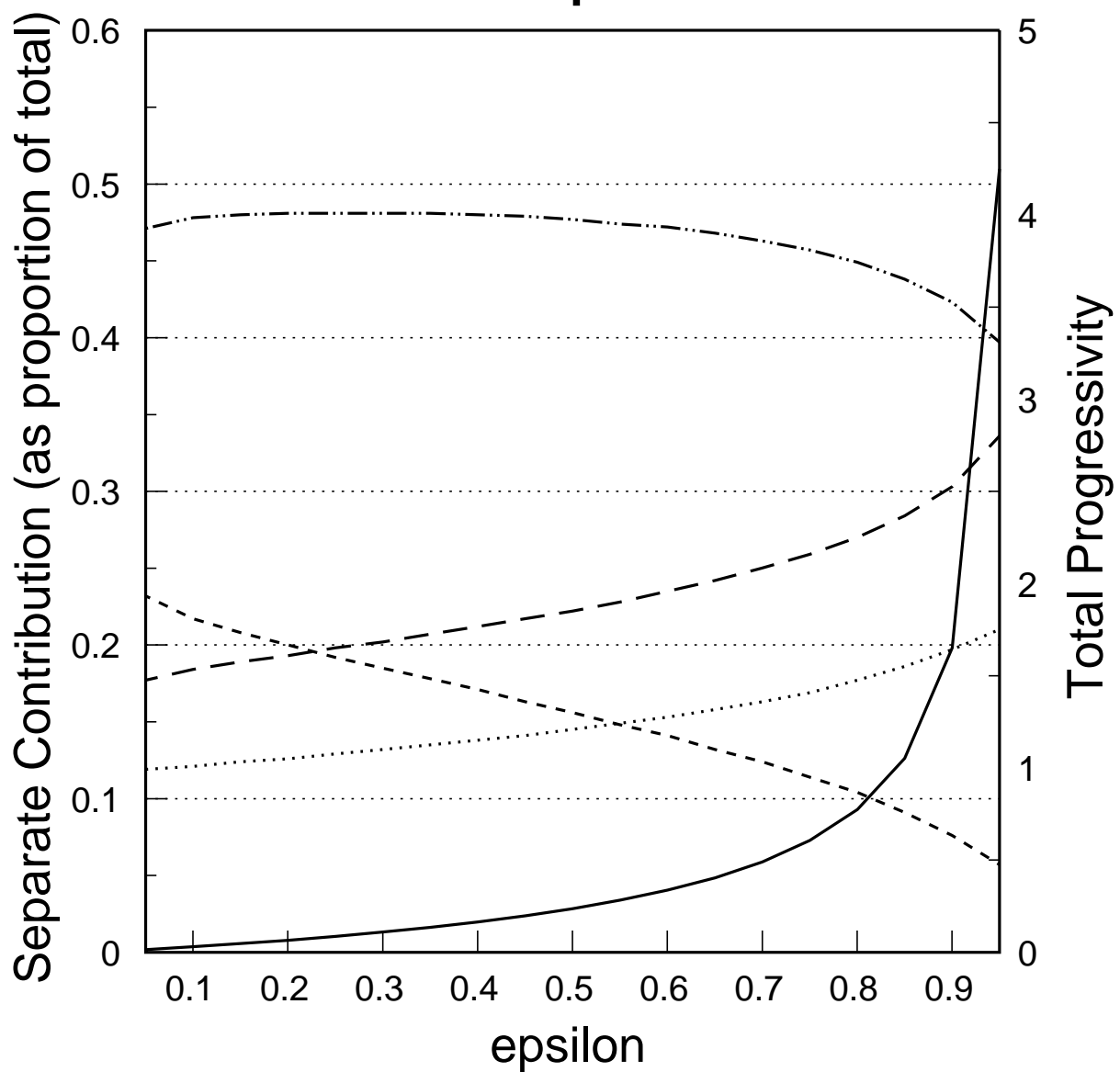
Increased Inequality and the Performance  
of a Progressive Tax





# Figure 2

## Tax Progressivity Over the Whole Population



A: Various benefits

B: Personal taxation

C: Family Supplement and  
Supplementary Benefit

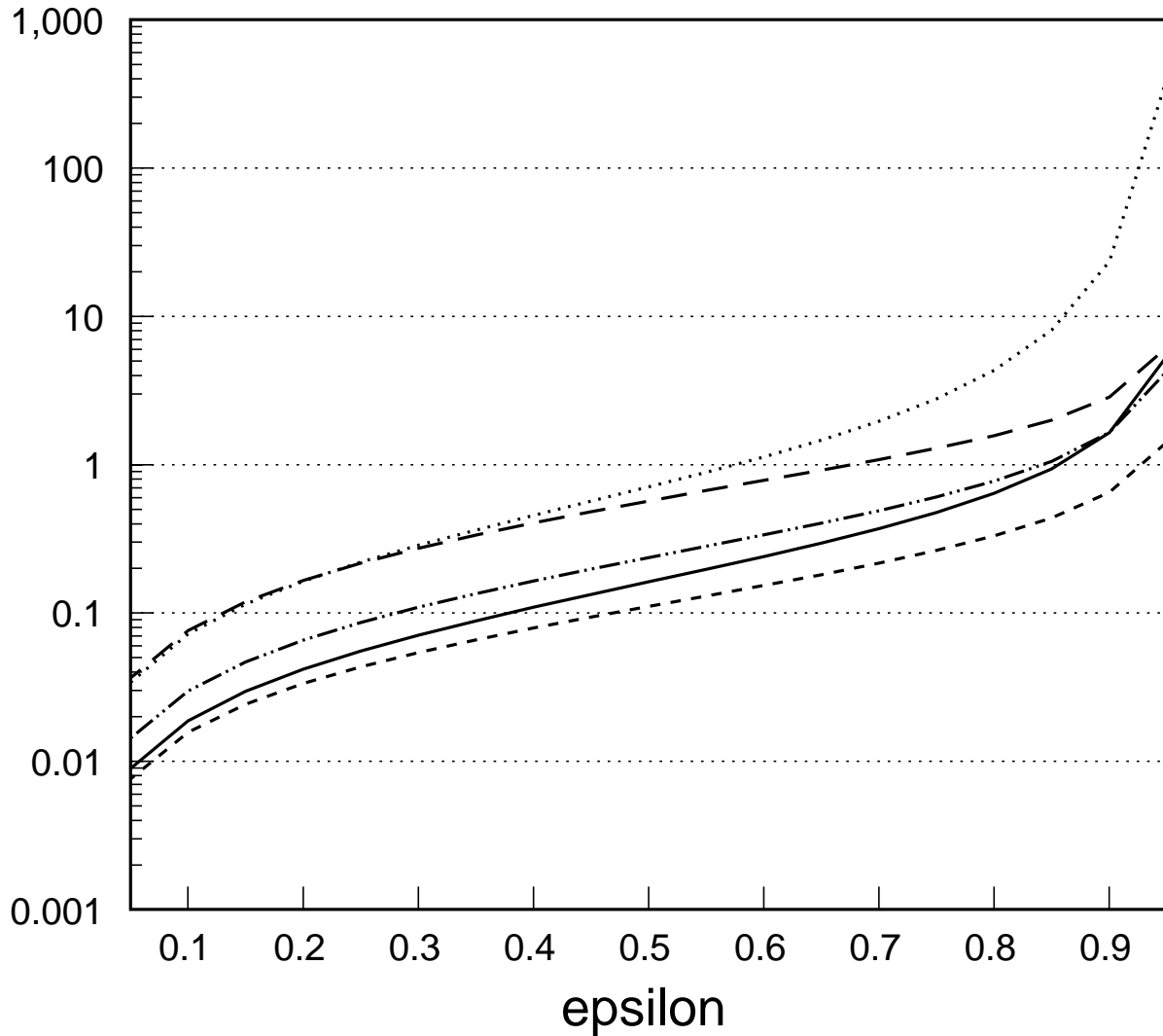
D: Housing dependent benefits

Total Progressivity

# Figure 3

## Progressivity Indices of Four Groups

log scale



Younger families  
without children

—

Two adults  
with children

- - - -

One adult  
with children

.....

Older families

- . - .

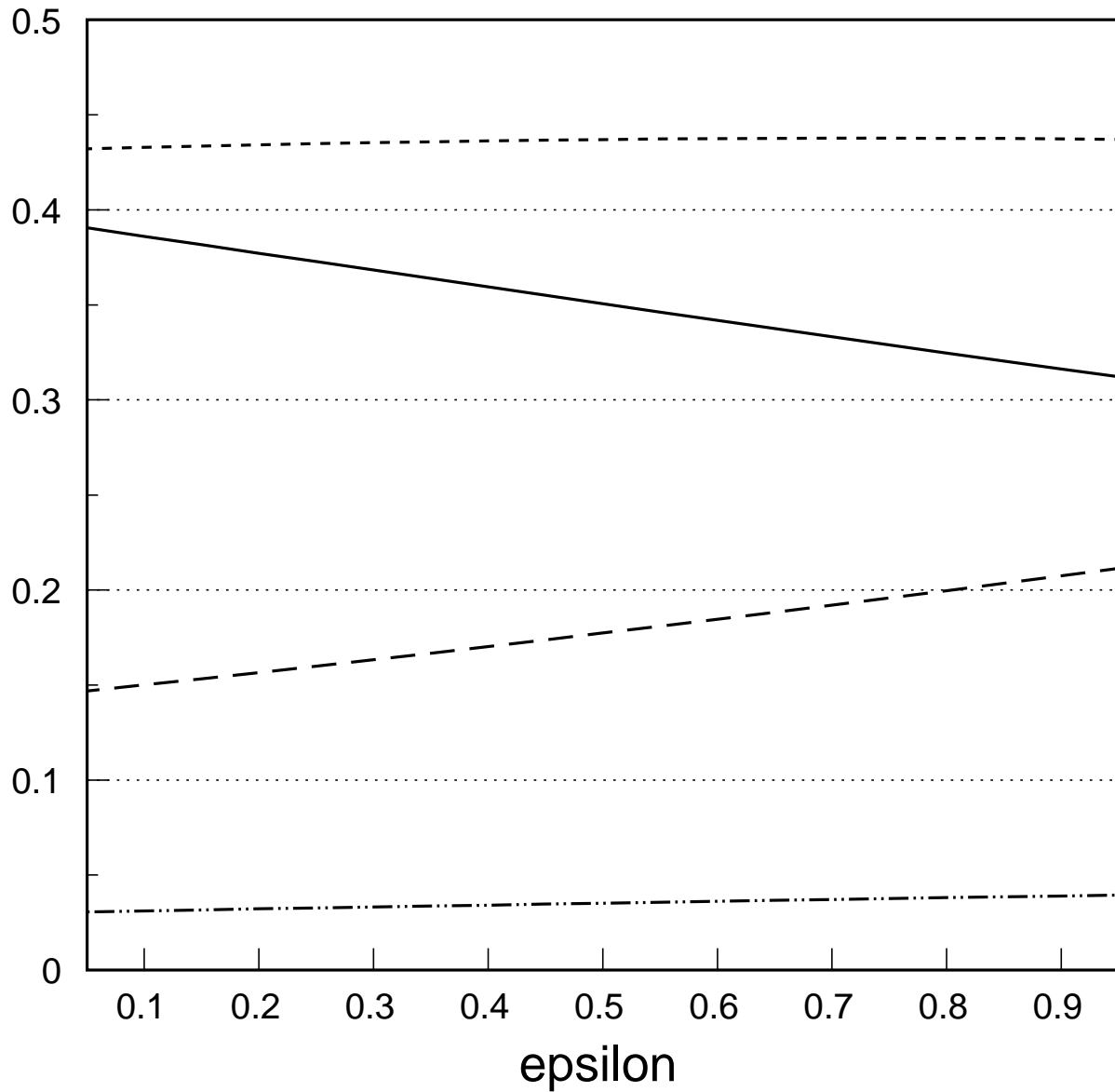
Whole population

.....

# Figure 4

## Group Contributions to Overall Progressivity

proportion of overall progressivity



Younger families without children    Two adults with children    One adult with children    Older families

—    - - - -    - . - . - .    - - - -